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Characterization of Pathos Adjacency Blict Graph of a Tree

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Abstract: In this paper we introduce the concept of pathos adjacency blict graph $PBn(T)$ of a tree T and present the characterization of graphs whose pathos adjacency blict graphs are planar, outerplanar, minimally non-outerplanar and Eulerian.

Key Words: Pathos, outerplanar, Smarandachely blict graph, crossing number cr(G), inner vertex number $i(G)$, minimally non-outerplanar.

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§1. Introduction

All graphs considered in this paper are finite and simple. For standard terminology and notation in graph theory, not specifically defined in this paper, the reader is referred to Harary [3]. The operation of forming a graph valued function of a graph G is probably the most interesting operation by which one graph is obtained from another. The concept of *pathos* of a graph G was introduced by Harary [4], as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The *path number* of a tree T is equal to k, where $2k$ is the number of odd degree vertices of T. Also, the end vertices of every path of any pathos of a tree T are of odd degree [2]. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

A *pathos vertex* is a vertex corresponding to a path P in any pathos and a *block vertex* is a vertex corresponding to a block(or an edge) of a tree T. The *edge degree* of an edge pq of a tree T is the sum of the degrees of p and q.

The *lict graph* (Here "*lict*" indicates *"line cut vertex"*) of a graph G [6], written $n(G)$, is the graph whose vertices are the edges and cut vertices of G, with two vertices of $n(G)$ adjacent whenever the corresponding edges of G are adjacent or the corresponding members of G are incident, where the edges and cut vertices of G are called its members. Let C be a block set of G. A *Smarandachely blict graph* $B^C(G)$ is the graph whose vertices are the edges, cut vertices

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and blocks in C, with two vertices of $Bn(G)$ adjacent whenever the corresponding members of G are adjacent or incident, where the edges, cut vertices and blocks in C are called its *members*. Particularly, if C is all blocks of G, such a $B^C(G)$ is called a *blict graph* (Here "*blict*" indicates "*block line cut vertex*") of a graph G [1], written by $Bn(G)$.

The *pathos line graph* of a tree T [1], written $PL(T)$, is the graph whose vertices are the edges and paths of pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T are adjacent and the edges that lie on the corresponding path P_i of pathos of T. The *pathos lict graph* of a tree T [1], written $Pn(T)$, is the graph whose vertices are the edges, cut vertices and paths of pathos of T, with two vertices of $Pn(T)$ adjacent whenever the corresponding edges of T are adjacent, edges that lie on the corresponding path P_i of pathos and the edges incident to the cut vertex of T.

A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be an *outerplanar*, *i.e.* $i(G) = 0$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing its outer planarity. A graph G is said to be *minimally non-outerplanar* if $i(G)$=1 [5]. The least number of edge-crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by cr(G).

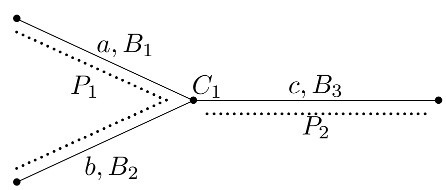

Figure 1 Tree T

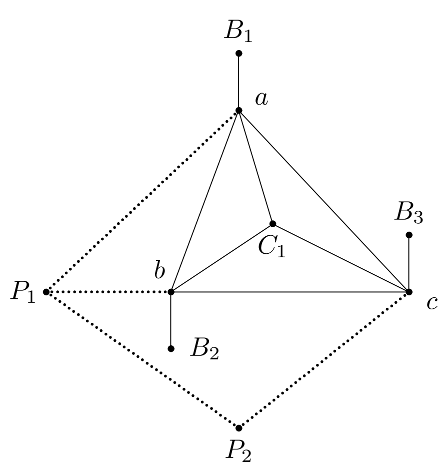

Figure 2 Pathos adjacency blict graph $PBn(T)$ of T

Definition 1.1 *The pathos adjacency blict graph of a tree T, written $PBn(T)$, is the graph whose vertices are the edges, paths of pathos, cut vertices and blocks of T, with two vertices of $PBn(T)$ adjacent whenever the corresponding edges of T are adjacent, edges that lie on the corresponding path P_i of pathos, edges incident to cut vertex and edges that lie on the blocks of T. Two distinct pathos vertices P_m and P_n are adjacent in $PBn(T)$ whenever the corresponding paths of pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex, say v_c in T.*

Since the pattern of pathos for a tree is not unique, the corresponding pathos adjacency blict graph is also not unique. Figure 1 shows a tree T and Figure 2 is its corresponding $PBn(T)$.

The following existing results are required to prove further results.

Theorem A([1]) *The pathos line graph $PL(T)$ of a tree T is planar if and only if $\Delta(T) \leq 4$.*

Theorem B([1]) *Let T be a tree on p vertices and $q = p-1$ edges such that d_i and C_j are the degrees of vertices and cut vertices C of T, respectively. Then the pathos lict graph $Pn(T)$ has $(q+k+C)$ vertices and $\frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j$ edges, where k is the path number of T.*

Theorem C([1]) *The blict graph $Bn(G)$ of a graph G is planar if and only if $\Delta(T) \leq 3$ and every vertex of degree three is a cut vertex.*

Theorem D([3]) *If G is a graph on p vertices and q edges, then $L(G)$ has q vertices and $-q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$ edges, where d_i is the degree of vertices of G.*

Theorem E([6]) *The lict graph $n(G)$ of a graph G is planar if and only if G is planar and the degree of each vertex is at most three.*

§2. Preliminary Results

Remark 2.1 For any tree T with $p \geq 3$ vertices, $L(T) \subseteq PL(T) \subseteq PBn(T)$, $L(T) \subseteq Bn(T) \subseteq PBn(T)$ and $L(T) \subseteq Pn(T) \subseteq PBn(T)$. Here $\subseteq$ is the subgraph notation.

Remark 2.2 If the edge degree of an edge pq in a tree T is even(odd) and p and q are the cut vertices, then the degree of the corresponding vertex pq in $PBn(T)$ is even(odd).

Remark 2.3 If the degree of an end edge(or pendant edge) in a tree T is even(odd), then the degree of the corresponding vertex in $PBn(T)$ is odd(even).

Remark 2.4 For any tree T (except star graph), the number of edges in $PBn(T)$ whose end vertices are the pathos vertices is given by $(k-1)$, where k is the path number of T.

Remark 2.5 If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges in $PBn(T)$ whose end vertices are the pathos vertices is given by $\frac{k(k-1)}{2}$, where k is the path number of T. For example, edge P_1P_2 in Figure 2.

Remark 2.6 Since every block vertex of $PBn(T)$ is an end vertex (For example, the block vertices B_1,B_2 and B_3 in *Figure 2*), $PBn(T)$ does not contain a spanning cycle. Hence it is always non-Hamiltonian.

§3. Lemmas

Here we present two simple lemmas on the graph $PBn(T)$.

Lemma 3.1 *Let T be a tree (except star graph) on p vertices and q edges such that d_i and C_j are the degrees of vertices and cut vertices C of T, respectively. Then $PBn(T)$ contains $(2q+k+C)$ vertices and*

$$\frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j + q + (k-1)$$

edges, where k is the path number of T.

Proof Let T be a tree (except star graph) on p vertices and q edges. By definition, the number of vertices in $PBn(T)$ equals the sum of number of edges, paths of pathos, cut vertices and the blocks of T. Since every edge of T is a block, $PBn(T)$ contains $(2q+k+C)$ vertices.

By Theorem B, the number of edges in $Pn(T)$ is $\frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j$. The number of edges in $PBn(T)$ equals the sum of edges of $Pn(T)$, edges that lie on the corresponding path P_i of pathos of T and the number of edges whose end vertices are the pathos vertices. By Remark 2.4, the number of edges in $PBn(T)$ is given by

$$\frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{j=1}^{C} C_j + q + (k-1). \qquad \square$$

Lemma 3.2 *Let T be a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges such that d_i and C_j are the degrees of vertices and cut vertex C of T, respectively. Then $PBn(T)$ contains $(2m+k+1)$ vertices and $\frac{1}{2}\sum_{i=1}^{n} d_i^2 + 2m + \frac{k(k-1)}{2}$ edges, where k is the path number of T.*

Proof Let T be a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges. Since T has exactly one cut vertex C, $PBn(T)$ contains $(2m+k+1)$ vertices. For a star graph T, the number of edges in $PBn(T)$ equals the sum of number of edges of $L(T)$, thrice the number of edges of T and the number of edges whose end vertices are the pathos vertices.

By Theorem D and Remark 2.5, we know that

$$-m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + 3m + \frac{k(k-1)}{2} = \frac{1}{2}\sum_{i=1}^{n} d_i^2 + 2m + \frac{k(k-1)}{2}.$$

Whence, we get the conclusion. $\square$

§4. Main Results

Theorem 4.1 *The pathos adjacency blict graph $PBn(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$, for every vertex $v \in T$.*

Proof Suppose $PBn(T)$ is planar. Assume that $\Delta(T) > 3$. If there exists a vertex p of degree 4 in T, by Theorem[A], $PL(T)$ is planar and contains K_4 as an induced subgraph. In $Pn(T)$, the vertex p is adjacent to every vertex of K_4. This gives K_5 as subgraph in $PBn(T)$. Clearly, $PBn(T)$ is nonplanar, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 If T is a path P_n on $n \geq 3$ vertices, then each block of $n(T)$ is K_3 and it has exactly $(n-2)$ blocks. The path number of T is exactly one and the corresponding pathos vertex is adjacent to at most two vertices of each block of $n(T)$. The pathos vertex together with each block of of $n(T)$ gives $(n-2)$ number of $\langle K_4 - e\rangle$ subgraphs in $Pn(T)$. Furthermore, every edge of T is a block. Hence the adjacency of block vertices and the vertices of $L(T)$ gives $(n-2)$ number of $\langle K_4 - e\rangle$ subgraphs in $PBn(T)$. Clearly, the crossing number of $PBn(T)$ is zero, *i.e.* cr($PBn(T)$)=0. Hence $PBn(T)$ is planar.

Case 2 Suppose that T is not a path such that $\Delta(T) \leq 3$. By Theorem E, $n(T)$ is planar. Moreover, each block of $n(T)$ is either K_3 or K_4. The path number of T is at least one and the corresponding pathos vertices are adjacent to at most two vertices of each block of $n(T)$. Hence $Pn(T)$ contains at least one copy of K_3 and K_4 as its subgraphs. Finally, on embedding $PBn(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, the crossing number of $PBn(T)$ becomes zero, *i.e.*, cr($PBn(T)$)=0. Hence $PBn(T)$ is planar. This completes the proof. □

Theorem 4.2 *The pathos adjacency blict graph $PBn(T)$ of a tree T is an outerplanar if and only if T is a path on P_n on $n \geq 3$ vertices.*

Proof Suppose $PBn(T)$ is an outerplanar. Assume that T has a vertex p of degree three. The edges incident to p and the cut vertex p gives K_4 as subgraph in $Pn(T)$. By Remark 2.1, the inner vertex number of $PBn(T)$ is non-zero, *i.e.* $i(PBn(T)) \neq 0$, a contradiction.

Conversely, suppose that T is a path P_n on $n \geq 3$ vertices. By Case 1 of Theorem 4.1, $PBn(T)$ contains $(n-2)$ number of $\langle K_4 - e\rangle$ as its subgraphs. Clearly, *i.e.* $i(PBn(T))$=0. Hence $PBn(T)$ is an outerplanar. This completes the proof. □

Theorem 4.3 *For any tree T, $PBn(T)$ is not maximal outerplanar.*

Proof By Theorem 4.2, $PBn(T)$ is an outerplanar if and only if T is a path P_n on $n \geq 3$ vertices. Suppose that T is a path P_n on $n \geq 3$ vertices with the edge set $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. By Case 1 of Theorem 4.1, $Pn(T)$ contains $(n-2)$ number of $\langle K_4 - e\rangle$ as its subgraphs. Moreover, each edge of T is a block. Hence by definition, block vertices and the vertices of $L(T)$ are adjacent in $PBn(T)$, which in turn forms $(n-1)$ number of end edges in $PBn(T)$. Finally, since the addition of an edge between the block vertices increases the inner vertex number of $PBn(T)$ by at least one, $PBn(T)$ is not maximal outerplanar. This completes the proof. $\square$

Theorem 4.4 *For any tree T, $PBn(T)$ is not minimally non-outerplanar.*

Proof Proof by contradiction. Suppose that $PBn(T)$ of a tree T is minimally non-outerplanar. We consider the following cases.

Case 1 Suppose that $\Delta(T) \leq 2$. By Theorem 4.2, $PBn(T)$ is an outerplanar, a contradiction.

Case 2 Suppose that $\Delta(T) \geq 3$.

We consider the following subcases of Case 2.

Subcase 2.1 Suppose that $\Delta(T) > 3$. By Theorem 4.1, $PBn(T)$ is nonplanar, a contradiction.

Subcase 2.2 Suppose that $\Delta(T) = 3$. Let p be a vertex of degree 3 in T. By Case 2 of Theorem 4.1, cr($PBn(T)$)=0, but it is easy to observe that (For example, the graph $PBn(T)$ in Figure 2) on embedding $PBn(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, the inner vertex number of $PBn(T)$ is at least two, *i.e.* $i(PBn(T)) \geq 2$. Hence $PBn(T)$ is not minimally non-outerplanar. This completes the proof. $\square$

Theorem 4.5 *For any tree T with $p \geq 3$ vertices, $PBn(T)$ is non-Eulerian.*

Proof Suppose that T is a tree with $p \geq 3$ vertices. Then there exists at least one cut vertex C of T which is incident to at least one end edge q or at least one block B. We consider the following two cases.

Case 1 If the degree of cut vertex C is odd, then the edge degree of q in T is even. By Remark 2.3, $PBn(T)$ contains odd degree vertex. Hence $PBn(T)$ is non-Eulerian.

Case 2 If the degree of cutvertex C is even, then the edge degree of q in T is odd. By Remark 2.3, $PBn(T)$ contains even degree vertex. But, since every edge of T is a block, degree of the corresponding block vertex in $PBn(T)$ is exactly one. Hence $PBn(T)$ is non-Eulerian. This completes the proof. $\square$

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